MODULE - 1

→ The word fluid is used to denote either a liquid or gas.

→ Study of dynamics of fluid is known as fluid dynamics. It is divided into three categories :-

1) hydrodynamics – flow of liquids.
2) Gas dynamics – flow of gases.
3) Aero dynamics – flow of air.

Generally aerodynamics is used to reffer other Two Types also, in practical applications aerodynamics may be applied to get the following objectives.

1) The prediction of forces and moments moving through a liquid.
2) Determination of flows moving internally through duct.

The first application is known as external aerodynamics.

The second one is internal aerodynamics

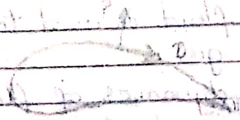
→ AERODYNAMIC FORCES.

Aerodynamic forces and moments in the body are due to only two basic sources

1) Pressure distribution over the surface

of the body.

2) Shear stress distribution over the body surface.



Lift force $L = \frac{1}{2}\rho V^2 S C_L$

Drag force, $D = \frac{1}{2}\rho V^2 S C_D$

Moment force, $M = \frac{1}{2}\rho V^2 S l C_M$

⇒ AERODYNAMIC PROBLEMS

Basic flow Equation ← Vector Algebra → using parameters

1) continuity equation
2) Momentum equation
3) Energy Equation

velocity, circulation etc

→ Solution

⇒ CONTINUITY EQUATION

Physical Principle : Mass is conserved

consider a control volume which is fixed in space. The fluid flows through it with velocity V. V is the control volume, dv is an elemental volume inside the control volume, dv is an elemental volume inside the control volume. Net Mass flow out of control volume through surface S (B) = time rate of decrease of mass inside the control volume (C)

$\therefore B = C \rightarrow (1)$

$\dot{m} = \rho A V = \rho V dS$

$B = \oiint_S \rho V dS \rightarrow (2)$ net mass flow out of the entire control surface S.

$m = \rho V$

$= \rho dV$

$= \iiint_V \rho dV$ → Total mass inside the control volume.

$m/t = \frac{\partial}{\partial t}\iiint_V \rho dV$

$$\boxed{C = -\frac{\partial}{\partial t}\iiint_V \rho dV}$$

$$C = -\frac{\partial}{\partial t}\iiint_V \rho dV$$

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Time rate of mass inside the volume

$$\boxed{C = \frac{-\partial}{\partial t} \iiint_V \rho \, dV} \quad \text{---(3)}$$

$B = C$ — becomes

$$\iint \rho v \, ds = \frac{-\partial}{\partial t} \iiint_V \rho \, dV$$

$$\iint_S \rho v \, ds + \frac{\partial}{\partial t} \iiint_V \rho \, dV = 0$$

This called continuity equation in integral form. It is one of the fundamental equation in fluid dynamics?

$$\iiint_V \frac{\partial \rho}{\partial t} dv + \iint_S \rho v \, ds = 0$$

from Divergence theorem

$$\boxed{\iint_S \rho \, ds = \iiint_V \nabla \cdot A \, dv}$$

$$\iiint_V \frac{\partial \rho}{\partial t} dv + \iiint_V \left(i\frac{\partial}{\partial x} + J\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \rho v \, dv = 0 \qquad \left[\nabla = i\frac{\partial}{\partial x} + J\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right]$$

$$\iiint_V \frac{\partial \rho}{\partial t} dv + \iiint_V \nabla \cdot \rho v \, dv = 0$$

$$\iiint_V \left[\frac{\partial \rho}{\partial t} + \nabla \rho v \right] dv = 0$$

$$\oiint_S A \cdot ds = \iiint_V \nabla \cdot A \, dv$$

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If the integrant is a finite number the integral over the part of the control volume be equal and opposite such that the net integration could be zero.

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla (\rho V) = 0}$$

Continuity equation in the form of partial differential equation.

In the case of ~~study~~ steady flow,

$$\frac{\partial \rho}{\partial t} = 0, \quad \nabla (\rho v) = 0$$

⇒ Scalar field.

A scalar quantity given as a function of coordinate space and time t is called. ~~the~~ scalar field.

ex : pressure, density and temperature.

Cartesian [$x\,y\,z$]
cylindrical [$r\,\theta\,z$]
spherical [$r\,\theta\,\phi$]

$$P = P_1(x, y, z, t) = P_2(r, \theta, z, t) = P_3(r, \theta, \phi, t)$$

$$T = T_1(x, y, z, t) = T_2(r, \theta, z, t) = T_3(r, \theta, \phi, t)$$

$$\rho = \rho_1(x, y, z, t) = \rho_2(r, \theta, z, t) = \rho_3(r, \theta, \phi, t)$$

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And these are the scalar fields for pressure and density respectively.

→ similarly a vector quantity given as function of co-ordinate space and time is called a vector field.

Ex velocity -

$V = V_x i + V_y j + V_z k$ — vector field for velocity in space

where $V_x = V_x(x, y, z, t)$

$V_y = V_y(x, y, z, t)$

$V_z = V_z(x, y, z, t)$

→ line integral

Consider a curve C in space connecting two points a and b. let ds be the elemental length of the curve and n be a unit vector tangent to the curve.

consider a vector field — $A = A(x, y, z) =$

$A = (r, \theta, z) = A(r, \theta, \phi)$

then $ds = n\,ds$

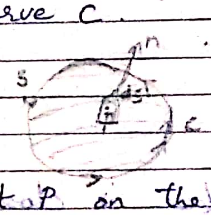
then the line integral of A on g curve C from point a to b is

$$\int_a^b A \cdot ds = \int_a A \cdot ds$$

If the curve is closed; then the line integral is given by, $\oint A \cdot ds$

→ SURFACE INTEGRAL.

consider an open surface S bounded by closed curve C.



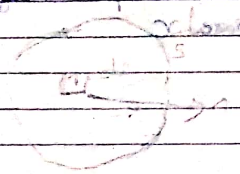
At point P on the surface, let ds be an elemental area of the surface and n be a unit vector normal to the surface, define a vector elemental area as $ds = n \times ds$ in terms of ds the surface integral over the surface S can be defined in three ways.

1. surface integral of a scalar p over open surface S. $\iint_S p.ds$, the result is a vector.

2. The surface integral of a vector A over the open surface S $\iint_S A.ds$, the result is scalar.

3 The surface integral of a vector A over the open surface S. $\iint_S A \times ds$, the result is a vector.

surface S is closed



If the surface is closed, n points out of the surface away from the enclosed volume. The surface integrals over the closed surfaces are

1. $\oiint_S P.ds$ 2. $\oiint_S A.ds$ 3. $\oiint_S A \times ds$

→ VOLUME INTEGRAL

Consider a volume V in the space let P be a scaler field in this space. volume integral over the volume V of the quantity P is written as $\iiint P\,dv$

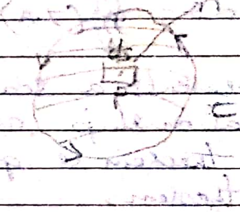
P volume integral of a scalar draw over the volume V, the result is a scalar.

Let A be a vector field in space the volume integral over the volume V of a quantity A is written as

$\iiint_V A\,dv$

Volume integral of a vector A over the volume V the result is a vector.

→ Relation b/w line surface and volume integrals: consider an open area S, bounded by a closed curve



Let A be a vector field, then the line integral of A over C is related to the surface integral of A over S by Stokes Theorem.

$$\boxed{\oint_C A \cdot ds = \iint_S (\nabla \times A) \cdot ds} \text{ — Stokes Theorem}$$

Consider the volume "V" enclosed by the surface S.

The surface and volume integrals of the vector field A are related to the divergence theorem

$$\oiint_S A \cdot ds = \iiint_V (\nabla \cdot A)\, dv.$$

If 'p' represents a scalar field a vector relationship analogous to divergence theorem is given by gradient theorem.

$$\oiint_S p\, ds = \iiint_V \nabla p\, dv.$$

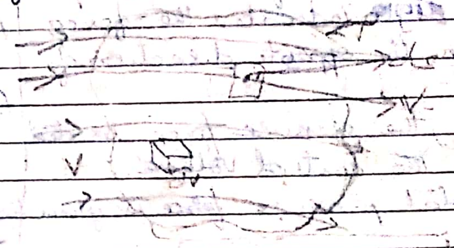
MOMENTUM EQUATION

Momentum equation ⇒ physical principle =

$$f = \frac{d}{dt}(mv)$$

Newton's 2nd law

Consider a finite control volume fixed in space with volume V and control surface S -



Forces are from two sources

1) Body force — Gravity, Electromagnetic forces, or any other forces which acts at a distances on the fluid inside volume V.

2) Surface forces — Pressure and shear stress acting on the control surface S.

Let f be the Net force per unit

mass then the body force on the elemental volume

dv. $\boxed{f = f \times \rho dv} \rightarrow (1)$

$f = \frac{F}{m}$
$m = \rho dv$
$F = f \times \rho dv$

$f = \frac{F}{m}$

$f = \frac{F}{\rho dV}$

$F = f \times \rho dV$

Total body force is given by

$$\boxed{F = \iiint_V f \times \rho \times dv} \rightarrow (2)$$

The Pressure force $F_p = -pds$

$p = \frac{F_p}{A}$
$p = \frac{F_p}{ds}$
$F_p = pds$

-ve sign indicates the force is in the opposite direction of ds.

i.e, the pressure force directed into the control volume.

Total pressure force is given by

$$\boxed{F_p = -\oiint_S p ds} \rightarrow (3)$$

In a viscous flow $f_{viscous}$ will be there

$$\text{Total force} = \iiint_V f \times \rho dv - \oiint_S pds + F_{viscous} \quad (4)$$

$\rho = \frac{m}{V}$, $m = \rho dV$

$u = 0 - 1$

Time rate of change of momentum is the sum of two terms.

(1) Net flow of momentum out of control volume across surface $S = (G) \rightarrow (5)$

(2) Time rate of change of momentum due to unsteady fluctuations of flow properties inside $V = H \rightarrow (6)$

Net flow of momentum out of control volume across the surface S is simply outflow minus inflow of momentum across control surface.

The mass flow rate across the elemental area ds, $\dot{m} = \rho A V = \rho V ds$

So the flow of momentum $= \rho V ds \times V$.

Net flow of momentum out of control volume through S, $G = \oiint (\rho V ds) V$

If G has a +ve value there is more momentum flowing out of control volume per second than flowing in and vice-versa.

The momentum of the fluid in the elemental volume $dv = \rho v dv$. (mv).

The momentum contained at any instant inside the control volume $= \iiint (\rho V) dv$

Time rate of change due to unsteady flow fluctuations, $H = \frac{\partial}{\partial t} \iiint_V (\rho V)\, dv$

~~Total forces~~ $\iiint_V f \times \rho \times dv - \oiint_S P\, ds + F_{viscous}$

$$= \oiint_S \rho V ds V + \iiint_V \frac{\partial \rho}{\partial t} V dv.$$

This is the momentum equation in integral form.

Apply gradient theorem $\oiint_S P ds = \iiint_V \nabla P dv$

$$* \iiint_V f \times \rho \times dv - \iiint_V \nabla P dv + F_{viscous} =$$

$$\oiint \rho V ds V + \iiint \frac{\partial \rho}{\partial t} V dv$$

Apply divergence theorem $\oiint_S \rho V ds = \iiint_V (\nabla \rho V V) dv$

$$\iiint_V f \times \rho dv - \iiint_V \nabla P dv + f_{viscous} = \iiint_V (\nabla \rho V V) dv + \iiint_V \frac{\partial \rho}{\partial t} V dv$$

Using cartesian coordinates:

$$V = ui + vj + wk$$

The 'x' component of the equation:

$$\iiint_V f_x \rho dv - \iiint_V \frac{\partial P}{\partial x} dv + (f_x)_{viscous} =$$

$$\oiint_S (\rho V ds) u + \iiint_V \frac{\partial \rho}{\partial t} u dv.$$

Apply divergence theorem:

$$\oiint_S \rho V ds = \iiint (\nabla \rho V u) dv.$$

$$\iiint_V f_x \rho dv - \iiint_V \frac{\partial P}{\partial x} dv + (f_x)_{viscous} = \iiint_V (\nabla \rho V u) dv + \iiint_V \frac{\partial \rho}{\partial t} u dv$$

$$\iiint_V \left[f_x \rho dv - \frac{\partial P}{\partial x} dv + (f_x)_{viscous} - (\nabla \rho V u) dv - \frac{\partial \rho}{\partial t} u dv \right] = 0$$

$$\iiint_V \left[f_x \rho - \frac{\partial P}{\partial x} - \nabla \rho V u - \frac{\partial \rho}{\partial t} u \right] dv = -(f_x)_{viscous}$$

$$f_x \rho - \frac{\partial P}{\partial x} - \nabla \rho V u - \frac{\partial \rho}{\partial t} u = -(f_x)_{viscous}$$

$$\rightarrow \frac{\partial P}{\partial x} + \nabla \rho V u + \frac{\partial \rho}{\partial t} u - f_x \rho - (f_x)_{viscous} = 0$$

$$\rightarrow \frac{\partial P}{\partial y} + \nabla \rho V v + \frac{\partial \rho}{\partial t} v - f_y \rho - (f_y)_{viscous} = 0$$

$$\rightarrow \frac{\partial P}{\partial z} + \nabla \rho V w + \frac{\partial \rho}{\partial t} w - f_z \rho - (f_z)_{viscous} = 0$$

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These equations are called Navier stoke's equations. This is momentum equation for a viscous flow.

* Assumptions:

(i) for steady flow $\frac{\partial \rho u}{\partial t} = 0$

(ii) For inviscous flow $(f_x)_{viscous} = 0$

(iii) No body force is acting, so $\rho f_x = 0$

So the equation become:

$$\nabla(\rho u V) = -\frac{\partial p}{\partial x};$$

$$\nabla(\rho v V) = -\frac{\partial p}{\partial y}$$

$$\nabla(\rho w V) = -\partial p/\partial z.$$

The momentum equation for an inviscous flow is called euler's equation.

* Streamline:

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Streamline is a curve whose tangent at any point is in the direction of velocity vector at that point. Let 'ds' is the element of the streamline and 'v' is the velocity. Streamline is mathematically expressed as:

$$ds \times V = 0$$

where $ds = dx\,i + dy\,j + dz\,k$

$V = ui + vj + \omega k$

$$ds \times V = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ u & v & \omega \end{vmatrix}$$

$$i(\omega dy - v dz) \pm j(\omega dx \mp u dz) + k(v dx - u\,dy) = 0$$

It's each component must be 0:

$\omega dy - v dz = 0$

$-\omega dx \pm u dz = 0$

$v dx - u dy = 0$

These are the differential equation for streamline.

* Derivation of momentum equation in terms of substantial derivative:

$$\nabla\cdot(\rho V) = \rho \Delta V + V \nabla \rho.$$

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from momentum equation:

$$\frac{\partial(\rho u)}{\partial t} + \nabla(\rho V u) + \frac{\partial P}{\partial x} - \rho f_x - (F_x)_{viscous} = 0 \quad \text{--- (1)}$$

The first term can be expressed as:

$$\frac{\partial(\rho u)}{\partial t} = \rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} \quad \text{--- (2)}$$

The second term $\nabla(\rho V u) = \nabla u(\rho V)$

$$= u\nabla(\rho V) + \rho V(\nabla u). \quad \text{--- (3)}$$

substitute eq (2) & (3) in eq (1):

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + u\nabla(\rho V) + \rho V(\nabla u) + \frac{\partial P}{\partial x} - \rho f_x - (F_x)_{viscous} = 0$$

$$\rho\frac{\partial u}{\partial t} + u\left[\frac{\partial \rho}{\partial t} + \nabla \rho V\right] + \rho V(\nabla u) + \frac{\partial P}{\partial x} - \rho f_x - (F_x)_{viscous} = 0.$$

$$\rho\frac{\partial u}{\partial t} + \rho V(\nabla u) + \frac{\partial P}{\partial x} - \rho f_x - (F_x)_{viscous} = 0.$$

$$\rho\frac{\partial u}{\partial t} + \rho V(\nabla u) = -\frac{\partial P}{\partial x} + \rho f_x + (F_x)_{viscous}$$

$$\rho\left[\frac{\partial u}{\partial t} + V(\nabla u)\right] = -\frac{\partial P}{\partial x} + \rho f_x + (F_x)_{viscous}$$

Substantial derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + V\nabla$

Substituting substantial derivative in above equation, we get:

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho f_x + (F_x)_{viscous}$$

Illly for y & z we get eqns:

$$\rho\frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \rho f_y + (F_y)_{viscous}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \rho f_z + (F_z)_{viscous}$$

This is the momentum equation in terms of substantial derivative.

* Incompressible Bernoulli's equation:

It is the famous equation in fluid dynamics. It is obtained by integrating the Euler's equation. Euler's equation establish the relationship b/w pressure, density and velocity.

* Assumptions:
(i) Flow is inviscous.
(ii) No body forces acts.
(iii) Flow is steady
(iv) Flow is incompressible.

Consider the x component of momentum equation in terms of substantial derivatives:

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho f_x + (F_x)_{viscous} \quad (1)$$

(When flow is inviscous $(F_x)_{viscous} = 0$
No body force acts, $\rho f_x = 0$.

Eq (1) becomes:

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} \quad (2)$$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Sub in (2)

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x}$$

Flow is steady, so $\frac{\partial u}{\partial t} = 0$.

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x}.$$

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial P}{\partial x}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \quad (3)$$

Multiply eq (3) with dx

$$u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx + w\frac{\partial u}{\partial z}dx = -\frac{1}{\rho}\frac{\partial P}{\partial x}dx \quad (4)$$

Consider the flow along a streamline.
Take the equation of streamline including x components:

$udz - wdx = 0$;
$vdx - udy = 0$;

$$\left.\begin{array}{l} udz = wdx \\ vdx = udy \end{array}\right\} \quad (5)$$

sub (5) in (4):

$$u\frac{\partial u}{\partial x}dx + udy\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial z}dz = -\frac{1}{\rho}\frac{\partial P}{\partial x}dx$$

$$u\left[\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right] = -\frac{1}{\rho}\frac{\partial P}{\partial x}dx$$

$$u.du = -\frac{1}{\rho}\frac{\partial P}{\partial x}dx \quad (6)$$

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udu can be written as: $\frac{1}{2}d(u^2) = \frac{1}{2}2udu = udu$

eq ⑥ becomes:

$$\frac{1}{2}d(u^2) = -\frac{1}{\rho}\frac{\partial P}{\partial x}dx \quad -⑦$$

IIIrly for y and z

$$\frac{1}{2}d(v^2) = -\frac{1}{\rho}\frac{\partial P}{\partial y}dy \quad -⑧$$

$$\frac{1}{2}d(w^2) = -\frac{1}{\rho}\frac{\partial P}{\partial z}dz \quad -⑨$$

Add ⑦, ⑧ & ⑨:

$$\frac{1}{2}(d(u^2) + \frac{1}{2}d(v^2) + \frac{1}{2}d(w^2)$$

$$= -\frac{1}{\rho}\frac{\partial P}{\partial x}dx - \frac{1}{\rho}\frac{\partial P}{\partial y}dy - \frac{1}{\rho}\frac{\partial P}{\partial z}dz$$

$$\frac{1}{2}\left[d(u^2) + d(v^2) + d(w^2)\right]$$

$$= -\frac{1}{\rho}\left[\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right]$$

$$\frac{1}{2}dV^2 = -\frac{1}{\rho}dP$$

$$\frac{1}{2}2VdV = -\frac{1}{\rho}dP$$

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$$dP = -\rho VdV$$

This is the Euler's equation. It applies to an inviscous flow (invicid) with no body forces and it relates the change in velocity along a streamline dV to the change in pressure dP along the same streamline.

For incompressible flow density is constant, Integrate the Euler's equation b/w 2 points (1 & 2) along streamline

$$\int_1^2 dP = -\int_1^2 \rho VdV$$

$$P_2 - P_1 = -\frac{\rho}{2}\left[V^2\right]_1^2$$

$$P_2 - P_1 = \frac{\rho}{2}\left[V_1^2 - V_2^2\right]$$

$$P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2$$

This is Bernoulli's equation.

It can be written as:

i.e., $P + \frac{1}{2}\rho V^2 =$ constant, along a streamline.

For rotational flow the value of constant will change from one streamline to another. In the case of irrotational flow the constant is same for all streamlines.

ie, $P + \frac{1}{2}\rho V^2 =$ constant throughout the flow.

The physical significance of the Bernoulli's equation is when pressure increases velocity decreases and vice versa.

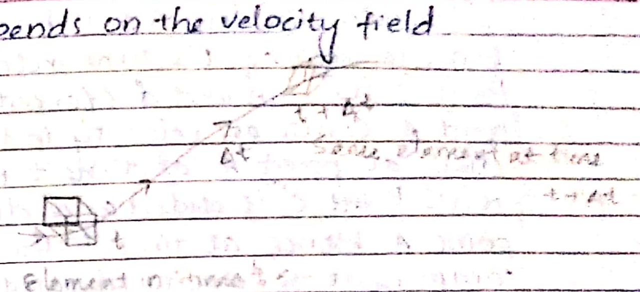
* Vorticity:

Vorticity is defined as the twice of the angular velocity. Denote vorticity by the vector ξ.

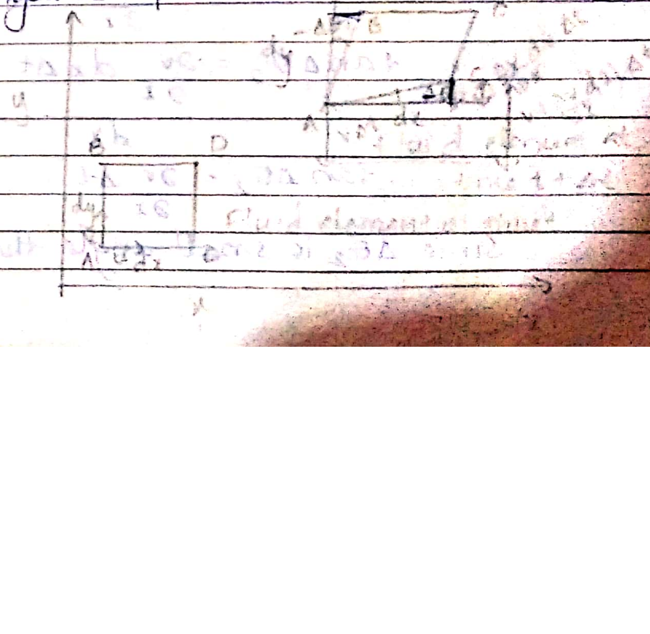
$$\xi = 2\omega$$

where ω is angular velocity.

Consider an infinitesimall fluid element moving in a flow field. As it translates along a streamline it may also rotate and in addition its shape may become distorted. The amount of rotation and distortion depends on the velocity field.



Consider a 2 dimensional flow in XY plane. Also consider an extremely small fluid element in this flow. Assume at the time 't', the shape of fluid element is rectangular. Assume that the fluid element is moving upward & to the right at the time $t + \Delta t$. During the time interval Δt the sides AB and AC have rotated through the angular displacement $-\Delta\theta_1$ and $\Delta\theta_2$.



Consider the line AC. It has rotated because during the time interval Δt. Point C has moved differently from point A. Consider velocity in the y-direction at point A at time 't' this velocity is 'v'. Point 'C' is a distance dx from point A. Hence at time 't' the vertical component of velocity of point 'C' is given by $v + \frac{\partial v}{\partial x} dx$. Hence distance in y-direction that 'A' moves during time increment $\Delta t = v\Delta t$. Distance in y-direction that 'C' moves during time increment $\Delta t = \left(v + \frac{\partial v}{\partial x} dx\right) \Delta t$

So, Net displacement in y-direction of 'C' relative to $A = \left(v + \frac{\partial v}{\partial x} dx\right) \Delta t - v\Delta t$

$$= \frac{\partial v}{\partial x} dx \, \Delta t$$

$$\tan \Delta\theta_2 = \frac{\frac{\partial v}{\partial x} dx \, \Delta t}{dx}$$

$$\tan \Delta\theta_2 = \frac{\partial v}{\partial x} \Delta t$$

Since $\Delta\theta_2$ is small angle then tan $\Delta\theta_2$ is approximately equal to $\Delta\theta_2$

i.e., $\Delta\theta_2 = \frac{\partial v}{\partial x} \Delta t$

$$\frac{\Delta\theta_2}{\Delta t} = \frac{\partial v}{\partial x} \quad \text{--- (1)}$$

Consider the line AB. It has rotated because during the time interval Δt. Point B has moved differently from point A. Consider velocity in the x-direction at point A at time t this velocity is 'u'. Point 'B' is a distance dy from point A. Hence at time 't' the horizontal component of velocity of point B is given by $u + \frac{\partial u}{\partial y} dy$. Hence distance in x direction that 'A' moves during time increment $\Delta t = u\Delta t$. Distance in x-direction that 'B' moves during time increment $\Delta t = \left(u + \frac{\partial u}{\partial y} dy\right) \Delta t$. So Net displacement in x-direction of 'B' relative to $A = \left(u + \frac{\partial u}{\partial y} dy\right)\Delta t - u\Delta t$

$$= \frac{\partial u}{\partial y} dy \, \Delta t$$

$$\tan(-\Delta\theta_1) = \frac{\frac{\partial u}{\partial y} dy \Delta t}{dy}$$

$$\tan(-\Delta\theta_1) = \frac{\partial u}{\partial y} \Delta t$$

$$-\Delta\theta_1 = \frac{\partial u}{\partial y} \Delta t$$

$$\frac{\Delta\theta_1}{\Delta t} = -\frac{\partial u}{\partial y} \quad \text{— (2)}$$

Consider the angular velocities of lines AB and AC defined as $\frac{d\theta_1}{dt}$ and $\frac{d\theta_2}{dt}$.

~~these are angular~~

$$\lim_{\Delta t \to 0} \frac{\Delta\theta_1}{\Delta t} = \frac{d\theta_1}{dt}$$

from eq (2):

$$\frac{d\theta_1}{dt} = -\frac{\partial u}{\partial y} \quad \text{— (3)}$$

From eq (1) $\lim_{\Delta t \to 0} \frac{\Delta\theta_2}{\Delta t} = \frac{d\theta_2}{dt}$

$$\frac{d\theta_2}{dt} = \frac{\partial v}{\partial x} \quad \text{— (4)}$$

Angular velocity of the fluid element in xy plane is the average of the angular velocities of line AB and AC. So angular velocity is in direction of z.

Let ω_z denotes the net angular velocity

$$\omega_z = \frac{\frac{d\theta_1}{dt} + \frac{d\theta_2}{dt}}{2} = \frac{1}{2}\left[-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$$

$$\omega_x = \frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]$$

$$\omega = \omega_x i + \omega_y j + \omega_z k$$

$$\omega = \frac{1}{2}\left[\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right] i + \left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right] j + \left[\frac{\partial v}{\partial x} - \partial u/\partial y\right] k\right]$$

This equation expresses the angular velocity of the fluid element in terms of derivatives of the velocity field.

vorticity $= 2\omega$

$$\xi = 2 \cdot \frac{1}{2}\left[\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right] i + \left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right] j + \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] k\right]$$

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$\xi = \nabla \times V$

In a velocity field the curl of the velocity is equal to vorticity.

There are 2 cases:

(i) If $\nabla \times V \neq 0$, the flow is called rotational. This implies that fluid element have a finite angular velocity.

(ii) If $\nabla \times V = 0$, the flow is called irrotational. This implies that fluid elements have no angular velocity. Their motion through space is a pure translation.

* Green's Lamma Theorem:

Green's theorem gives a relationship b/w the line integral of a 2d vector field over a closed path in the plane and the double integral over the region it encloses. Sometimes green's theorem is used to transform a line integral into a double integral and vice versa.

Let 'C' be piecewise smooth simple closed cure in the plane. Let this smooth curve is enclosed in the region 'R' and assume

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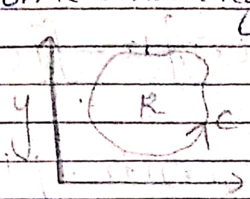
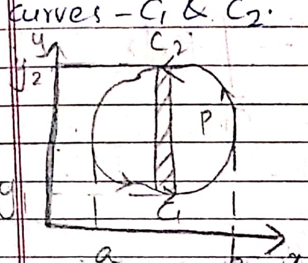
that 'P' and 'Q' and their first partial derivatives are continuous at each point in the region 'R' containing 'C'.



Let the curve 'C' is made up of 2 curves $- C_1$ & C_2.



For curve C_1:

$y_1 = f_1(x), \; a \leq x \leq b$

For curve C_2

$y_2 = f_2(x), \; b \leq x \leq a$

Integrating $\frac{\partial P(x,y)}{\partial y}$ w.r.t dy

$$\int_a^b \int_{y_1 = f_1(x)}^{y_2 = f_2(x)} \frac{\partial P(x,y)}{\partial y} \, dy \, dx$$

$$\int_a^b \Big[P(x,y) \Big]_{f_1(x)}^{f_2(x)} dx$$

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$$\int_a^b \left[P(x,f_2(x)) - P(x,f_1(x))\right] dx$$

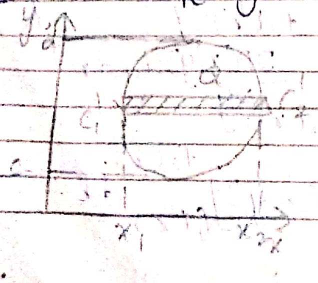
$$\int_a^b \left[P[x,f_2(x)]\right] dx - \int_a^b \left[P(x,f_1(x))\right] dx$$

$$-\int_b^a P(x,f_2(x))\,dx - \int_a^b P(x,f_1(x))\,dx$$

$$= -\int_{C_2} P\,dx - \int_{C_1} P\,dx$$

$$= -\oint_C P\,dx$$

$$\therefore \oint_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\,dy\,dx.$$



for curve $C_1 \Rightarrow x_1 = f_1(y)$ & for curve C_2 $x_2 = f_2(y)$.

~~for curve $C_2 \Rightarrow x_2 = f_2(x)$~~

for curve $C_1' \Rightarrow$, $d \le y \le c$.

for curve $C_2' \Rightarrow c \le y \le d$.

Integrating $\frac{\partial Q(x,y)}{\partial x}$ w.r.t dx.

$$\int_c^d \int_{x_1=f_1(y)}^{x_2=f_2(y)} \frac{\partial Q(x,y)}{\partial x}\,dx\,dy.$$

$$\int_c^d \left[Q(x,y)\right]_{f_1(y)}^{f_2(y)} dy.$$

$$\int_c^d \left[Q(f_2(y),y) - Q(f_1(y),y)\right] dy.$$

$$= \int_c^d Q(f_2(y),y)\,dy - \int_c^d Q(f_1(y),y)\,dy.$$

$$\cancel{-\int_d^c Q(f_2(y),y)\,dy - \int_c^d Q(f_1(y),y)\,dy}$$

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$$\int_{C_2} Q\,dy + \int_{C_1} Q\,dy = \oint_C Q\,dy$$

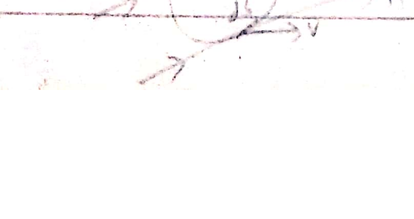
$$\oint_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\,dx\,dy$$

$$\therefore \oint_C P\,dx + \oint_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\,dx\,dy - \iint_R \frac{\partial P}{\partial y}\,dx\,dy$$

$$= \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy$$

* Circulation:

Consider a closed curve 'C' in a flow field. Let 'V' and 'ds' be the velocity and line segment respectively.



The circulation denoted by Γ (gamma) and is defined as:

$$\boxed{\Gamma = -\oint V\,ds} \quad \text{—} (1)$$

The circulation is the -ve of the line integral of velocity around the closed curve in the flow. It is a kinematic property depending only on the velocity.

Applying stoke's theorem on eq no (1):

$$\Gamma = -\iint_S (\nabla \times V)\,ds.$$

The circulation about a curve 'C' = the vorticity integrated over any open surface bounded by 'c'. If the flow is irrotational $\nabla \times V = 0$, then circulation is also equal to zero.

$$d\Gamma = -d\iint_S (\nabla \times V)\,ds$$

$$d\Gamma = -(\nabla \times V)\,ds\,n$$

If the curve 'C' shrinked to an infinitesimall size, then the circulation is $d\Gamma$.

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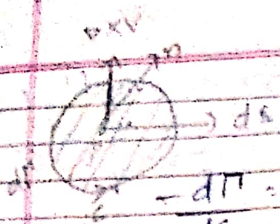

$$-\frac{d\Gamma}{ds} = (\nabla \times V).n$$

Vorticity normal to ds is equal to the -ve of the circulation per unit area, where the circulation is taken around the boundary of ds.

* Relationship b/w vorticity and Circulation

Vorticity	Circulation
(i) Vorticity Measurement is based on microscopic approach of the rotation of the fluid element.	(i) Circulation Measurement is based on macroscopic approach of the rotation of the fluid element.
(ii) Mathematically, vorticity can be defined as the curl of velocity field.	(ii) Mathematically, circulation can be defined as the line integral of velocity field around the closed curve.